ARTICLES

Phonon-assisted Boltzmann kinetics of a Bose gas: Generic solution for $T \leqslant T_c$

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The relaxation kinetics of an ideal Bose gas coupled to a phonon bath at temperatures below or equal to the critical temperature T_c is given within a unique scenario. During the first transient stage, which lasts a few characteristic scattering times, the initial distribution disappears. The following nonexponential relaxation towards a quantum degenerate equilibrium state with a Bose-Einstein condensate is a slow adiabatic process. For this second kinetic stage we find an *analytic generic solution* of the Boltzmann equation. The generic solution is independent of the initial distribution and completely defined by the ratio T_c/T_b of the critical and bath temperatures. [S1063-651X(97)01906-5]

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I. INTRODUCTION

More than six decades ago, the Boltzmann kinetic equation was generalized to include quantum statistics [1]. Because the Boltzmann kinetics involves all of the microscopic degrees of freedom simultaneously, an arbitrary initial distribution of particles usually evolves to the final (quasi)equilibrium state within a few characteristic scattering times $\tau_{\rm sc}$. This general conclusion is valid for classical Boltzmann gases [2], fermion systems [3], and boson systems at $T \gg T_c$ [4]. However, the relaxation kinetics of Bose particles at $T \leqslant T_c$ represents an important exception to this rule. The need to accumulate a macroscopic number of Bose particles in the ground-state mode (Bose-Einstein condensate) makes the kinetics rather unusual.

In this paper, we analyze the phonon-assisted relaxation kinetics of an ideal Bose gas at $T \leqslant T_c$. This problem has been approached for two rather different realizations, namely, the relaxation kinetics in the presence of a continuously acting external source of bosons [5–8] or from a given initial distribution [4,9]. In the first case, the final steady state of an open thermodynamic system with a possible condensate can indeed be reached within a few scattering times $\tau_{\rm sc}$. However, for a thermodynamically closed system of bosons coupled to a bath, the conclusions are controversial. Bose-Einstein condensation (BEC) in the time scale of $\tau_{\rm sc}$ has been found in Ref. [9] using a Fokker-Planck equation, while numerical simulations [4] show a considerable slowing down of relaxation kinetics at $T \leqslant T_c$. Our main result, i.e., the analytic generic solution for the phonon-assisted relaxation kinetics of a closed Bose system at $T \leqslant T_c$, clearly indicates the existence of the slow adiabatic stage. Here we have a unique example of the fundamental solution of the Boltzmann kinetic equation that is insensitive to an initial distribution at $t=0$.

A gas of excitons in semiconductors becomes quantum statistically degenerate at relatively high temperatures $T \sim T_c$, due to the small exciton translational mass of the order of the free-electron mass [10]. For example, an optically controlled Bose-Einstein statistics of orthoexcitons in Cu_2O has been reported [11]. Moreover, a possible Bose-Einstein condensation of paraexcitons in Cu_2O has been examined recently, both experimentally [12] and theoretically [13]. Simple estimates show that due to the small exciton radius in Cu_2O, the exciton-phonon interaction strongly dominates over the exciton-exciton scattering for concentrations $n_0 \leqslant 10^{17}$ cm^{-3} ($T_c \leqslant 3$ K). In the following analysis, we use the notation x in subscripts for a Bose particle exciton, taking into account a possible reference to excitons in Cu_2O.

The Boltzmann equation, which describes the kinetics of a spatially homogeneous system of excitons (x) coupled to a bath of acoustic phonons (ph), is given by

$$\frac{\partial}{\partial t} N_{\mathbf{k}} = -\frac{2\pi}{\hbar^2} \sum_{\mathbf{p}} |M_{x\text{-ph}}(\mathbf{p}-\mathbf{k})|^2 \{ [N_{\mathbf{k}}(1+n^{\rm ph}_{\mathbf{k}-\mathbf{p}})(1+N_{\mathbf{p}}) - (1+N_{\mathbf{k}}) n^{\rm ph}_{\mathbf{k}-\mathbf{p}} N_{\mathbf{p}}] \delta(e_{\mathbf{k}} - e_{\mathbf{p}} - \hbar\omega_{\mathbf{k}-\mathbf{p}}) + [N_{\mathbf{k}} n^{\rm ph}_{\mathbf{p}-\mathbf{k}}(1+N_{\mathbf{p}}) - (1+N_{\mathbf{k}})(1+n^{\rm ph}_{\mathbf{p}-\mathbf{k}}) N_{\mathbf{p}}] \times \delta(e_{\mathbf{k}} - e_{\mathbf{p}} + \hbar\omega_{\mathbf{p}-\mathbf{k}}) \}, \quad (1)$$

where $e_{\mathbf{k}} = \hbar^2 k^2 / 2M_x$ and $\hbar\omega_{\mathbf{p}-\mathbf{k}} = \hbar v_s |\mathbf{p}-\mathbf{k}|$ are the exciton and phonon energies, respectively; $N_{\mathbf{k}}$ and $n^{\rm ph}_{\mathbf{p}-\mathbf{k}}$ are the exciton and phonon occupation numbers, respectively; $M_{x\text{-ph}}(\mathbf{p}-\mathbf{k})$ is the matrix element of the exciton-phonon deformation potential interaction. The exciton-phonon coupling is determined by $|M_{x-\rm ph}(\mathbf{p}-\mathbf{k})|^2 = \hbar D^2 |\mathbf{p}-\mathbf{k}| / 2V\rho v_s$, where D is the deformation potential, V is the crystal volume, ρ is the crystal density, and v_s is the sound velocity. The first (second) term in the square brackets on the right-hand side (rhs) of Eq. (1) is due to the Stokes (anti-Stokes) scattering of excitons from the mode $\mathbf{k}$.

For an isotropic initial distribution of excitons in momentum space, the kinetic equation (1) reduces to the following equation for the exciton distribution function N_e in the one-dimensional energy space:

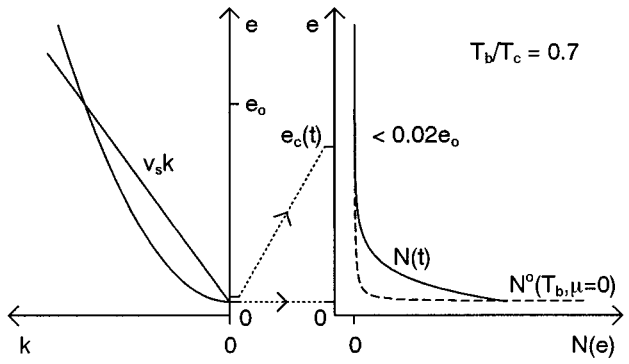

FIG. 1. Energy diagram for the phonon-assisted relaxation into the ground-state mode $e=0$ (left) and the distribution $N_e=N_e(\tau=226\tau_{\rm sc})$ in the adiabatic stage compared to the equilibrium distribution (dashed line) (right). Because of $e_0>k_BT_b$, $N^{\rm eq}_{e_0}\ll 1$ and the quantum degenerate statistics of excitons occurs at $e\ll e_0$.

$$\frac{\partial}{\partial t}N_e=-\left(\frac{\sqrt{M_x}D^2}{4\sqrt{2}\pi\hbar^4v_s^4\rho\sqrt{e}}\right)\int de_1(e-e_1)^2\{[N_e(1+n^{\rm ph}_{e-e_1})\times(1+N_{e_1})-(1+N_e)n^{\rm ph}_{e-e_1}N_{e_1}]\Theta(q_S-\sqrt{e}+\sqrt{e_1})\times\Theta(-q_S+\sqrt{e}+\sqrt{e_1})\Theta(e-e_1)+[N_en^{\rm ph}_{e_1-e}(1+N_{e_1})-(1+N_e)\times(1+n^{\rm ph}_{e_1-e})N_{e_1}]\Theta(q_{\rm AS}-\sqrt{e_1}+\sqrt{e})\times\Theta(-q_{\rm AS}+\sqrt{e_1}+\sqrt{e})\Theta(e_1-e)\},\tag{2}$$

where Θ is the Heaviside function and $q_{\rm S|AS}=\pm(e-e_1)/\sqrt{2M_x}v_s$ refers to the momentum transfer of Stokes and anti-Stokes scattering. In order to derive Eq. (2) from Eq. (1) one uses polar coordinates in momentum space, i.e., $(p_x,p_y,p_z)\to(p,\phi,\theta)$ with p_z along $\mathbf{k}$, performs the integration over $d\phi$ and $d\theta$, and makes the substitution $N_p=N_{\epsilon=\epsilon_p}$. The phonon system is supposed to be in equilibrium at the bath temperature T_b: $n^{\rm ph}_{\hbar\omega}=1/\exp(\hbar\omega/T_b)-1$ setting $k_B=1$.

We define the characteristic scattering time $\tau_{\rm sc}$ by

$$\tau_{\rm sc}=\left(\frac{\pi\hbar^4\rho}{4D^2M_x^3v_s}\right)[\exp(e_0/T_b)-1]\tag{3}$$

through the probability of anti-Stokes scattering from the ground-state mode $k=0$ (in energy space this mode corresponds to $e=0$) at $T=T_b$. Here $e_0=2M_xv_s^2$ is the energy of the partner state coupled to the mode $k=0$ by the resonant emission or absorption of an acoustic phonon with momentum p_0 (see Fig. 1). This partner state is determined by the energy-momentum conservation $e_0=\hbar^2p_0^2/2M_x=\hbar p_0v_s$.

According to Eq. (2), the phonon-assisted occupation of the ground-state mode $e=0$ is given by

$$\frac{\partial}{\partial t}N_{e=0}=\left(\frac{4D^2M_x^3v_s}{\pi\hbar^4\rho}\right)[N_{e_0}(1+n^{\rm ph}_{e_0})-N_{e=0}(n^{\rm ph}_{e_0}-N_{e_0})].\tag{4}$$

Here the first term in the square brackets on the rhs describes the population of the ground-state mode due to phonon-assisted spontaneous emission from the partner state e_0, while the second one characterizes the stimulated kinetics proportional to the occupation number $N_{e=0}$. Because $N_{e=0}\to\infty$ in the thermodynamic limit for $T_b\leqslant T_c$, the stimulated kinetics of Eq. (4) is crucial for the development of BEC of excitons. As recognized for the first time by Fröhlich for biological systems [5], the phonon-assisted relaxation kinetics in the presence of a continuously acting external source of bosons allows us to involve effectively the stimulated population of the ground-state mode. In this case, one keeps $n^{\rm ph}_{e_0}-N_{e_0}<0$ and stimulated BEC develops within a few scattering times $\tau_{\rm sc}$. This general scheme has been subsequently explored for excitonic systems [7,8].

In the present work, we analyze the phonon-assisted relaxation kinetics from a given initial distribution of excitons. This case corresponds to the experimental conditions of Ref. [12]. For $T_b\gg T_c$ an arbitrary initial distribution $N_e(t=0)$ relaxes within a few scattering times to a state with maximum entropy. According to the H theorem, the maximum entropy is realized for $N_e=1/\exp[(e-\mu)/T]-1$, with an effective chemical potential μ and an effective temperature T [14]. In equilibrium, $T=T_b$, $\mu=\mu(n_0,T_b)$ and the collision integral vanishes. However, for $T_b\leqslant T_c$ an anomalously slow, nonexponential kinetics occurs for the excitons at lowest energies. The ground-state mode $e=0$ couples with the relatively high-energy partner state $e_0>T_b$ only through the anti-Stokes scattering (see Fig. 1). Physically, a ''bottleneck'' relaxation into the ground-state mode $e=0$ occurs because N_{e_0} very quickly approaches $n^{\rm ph}_{e_0}$ during the thermalization and suppresses the stimulated processes. The collision integral on the rhs of Eq. (4) with $N_{e_0}\to n^{\rm ph}_{e_0}$ prevents an effective accumulation of a macroscopic number of excitons at $e=0$ and gives rise to the slow adiabatic stage of relaxation. Furthermore, during the adiabatic kinetics only the low-energy excitons are far from equilibrium, i.e., without a condensate fraction, while a high-energy tail of the distribution of excitons is in quasiequilibrium with the effective chemical potential $\mu=0$ and the effective temperature $T=T_b+\delta T$ $(1\gg\delta T/T_b>0)$.

In Sec. II we discuss a fundamental scenario of the phonon-assisted relaxation kinetics at $T\leqslant T_c$. The kinetics consists of a transient stage, which follows an initial distribution and lasts a few $\tau_{\rm sc}$, and the following slow adiabatic stage. For the adiabatic stage, we construct an analytic generic solution of the phonon-assisted Boltzmann kinetics.

In Sec. III we test the generic solution by straightforward numerical evaluation of the kinetc Eq. (2). The occupation kinetics of the ground-state mode $e=0$ is described analytically. We also discuss how the generic solution approaches BEC of excitons.

II. GENERIC SOLUTION OF BOLTZMANN KINETICS AT $T\leqslant T_c$

The bottleneck relaxation into the ground-state mode $e=0$, together with the need to accumulate at this mode a

macroscopic number of excitons, is responsible for the slowing down of the relaxation kinetics at $T_b \leqslant T_c$. The low-energy states $e \leqslant e_0/4 = M_x v_s^2/2$ couple only through anti-Stokes scattering with the corresponding phonon-separated partner modes $e \simeq e_0$ (see Fig. 1). Moreover, the bottleneck relaxation takes place for the all low-energy states $e \ll T_b$. Similarly to the ground-state mode $e=0$, for these modes $N_e^{\rm eq} \simeq T_b/e \gg 1$, while the anti-Stokes partner states have the high energies $\sim e_0 > T_b$ to be nondegenerate and to suppress the stimulated relaxation due to $N_{e \simeq e_0} \to n^{\rm ph}_{e \simeq e_0}$.

The high-energy states with nondegenerate statistics $N_e^{\rm eq} \ll 1$ equilibrate first. After a few scattering times, the low-energy ($e \leqslant e_c \ll T_b$) and the high-energy ($e > e_c$) excitons form two subsystems weakly interacting through phonon-mediated scattering. The H theorem [14], applied to the high-energy excitons, yields a quasiequilibrium Planck distribution $N_{e>e_c} = 1/[\exp(e/T)-1]$ with effective temperature $T = T_b + \delta T(t)$. The effective chemical potential $\delta\mu$ of the quasiequilibrated high-energy excitons can be neglected because $-\delta\mu \simeq [9\zeta^2(3/2)/16\pi T_c](\delta T)^2 \ll \delta T$ for $T \simeq T_c$ (in comparison, $\delta\mu \propto \delta T$ for $T \gg T_c$). Here $\zeta(3/2) \simeq 2.612$ is the Riemann zeta function. At this adiabatic stage of the relaxation, the low-energy subsystem shrinks in energy space $[e_c(t) \to 0]$, while the effective temperature T of the high-energy excitons approaches the bath temperature T_b $[\delta T(t) \to 0]$.

The absence of the direct phonon-mediated interaction between the low-energy excitons results in their slow adiabatic equilibration through the resonant coupling with the quasiequilibrium states $e \sim e_0$. It will be shown below that the proposed scenario of the phonon-assisted relaxation kinetics is justified by the numerical evaluation of the initial kinetic equation (2).

After the first transient relaxation, i.e., at $t \geqslant \Delta t_{\rm tr}$, the distribution $N_e(t)$ is still far from equilibrium for energies $e \leqslant e_c$, where $e_c = e_c(t) \ll T_b$ is the time-dependent critical boundary of this nonequilibrium region (see Fig. 1). For these low-energy excitons with $e \leqslant e_c$ Eq. (2) reduces to

$$\frac{\partial}{\partial \tau} N_\varepsilon = 1 - N_\varepsilon \left(\varepsilon - \frac{\delta T(\tau) e_0}{T_b^2} \right), \tag{5}$$

where $\varepsilon = e/T_b$ and $\tau = t[1-\exp(-e_0/T_b)]^{-1} \tau_{\rm sc}^{-1} \simeq t/\tau_{\rm sc} > 1$ are the dimensionless energy and time, respectively. The first term on the rhs of the reduced kinetic equation (5) stems from resonant phonon-assisted spontaneous emission to the state ε, while the second one describes the stimulated kinetics proportional to the distribution N_ε. For $\delta T = 0$, Eq. (5) yields a true equilibrium distribution $N_\varepsilon = 1/\varepsilon$ of the noncondensate excitons ($\varepsilon > 0$), i.e., Planck's function $N_\varepsilon^{\rm eq} = 1/[\exp(\varepsilon)-1]$ for $\varepsilon \leqslant \varepsilon_c(\tau) = e_c(\tau)/T_b \ll 1$. The temperature law $\delta T = \delta T(\tau)$ of the adiabatic cooling of the high-energy excitons is still unknown.

The reduced kinetic equation (5) is a linear differential equation for the nonequilibrium distribution $N_\varepsilon(\tau)$ of the low-energy excitons. Formally, the complete solution $N_\varepsilon(\tau)$ of Eq. (5) can be written as a sum of the homogeneous and inhomogeneous contributions

$$N_\varepsilon(\tau) = \tau \int_0^1 e^{\rho(\tau) - \rho(u\tau)} e^{-\tau\varepsilon(1-u)} du + N_\varepsilon^{\rm hom}(\tau). \tag{6}$$

Here the first term on the rhs is the inhomogeneous solution $N_\varepsilon^{\rm inhom}(\tau)$ and the homogeneous solution is given by

$$N_\varepsilon^{\rm hom}(\tau) = C_\varepsilon e^{\rho(\tau) - \varepsilon\tau}, \tag{7}$$

where C_ε are the integration constants independent of the dimensionless time τ. In Eqs. (6) and (7), the function $\rho(\tau) = (e_0/T_b^2) \int^\tau \delta T(\tau') d\tau'$ determines the temperature law of the adiabatic stage

$$\delta T(\tau) = \left(\frac{T_b^2}{e_0} \right) \frac{\partial}{\partial \tau} \rho(\tau). \tag{8}$$

The adiabatic stage of relaxation is mainly determined by the inhomogeneous solution $N_\varepsilon^{\rm inhom}(\tau)$ of Eq. (5), i.e., by the first term on the rhs of Eq. (6). The homogeneous solution $N_\varepsilon^{\rm hom}(\tau)$, which refers to the very beginning of the adiabatic stage at $\tau \simeq \tau_{\rm tr} = \Delta t_{\rm tr}/\tau_{\rm sc} \geqslant 1$ through the integration constants C_ε, decays with respect to $N_\varepsilon^{\rm inhom}(\tau)$. It will be shown after finding of the generic solution of Eq. (5) that indeed $N_\varepsilon^{\rm hom}(\tau)/N_\varepsilon^{\rm inhom}(\tau)|_{\tau \to \infty} \to 0$. Physically, this denotes that during the slow adiabatic stage, which lasts a great number of $\tau_{\rm sc}$, a system of excitons loses correlations with the initial distribution $N_e(t=0)$.

The conservation of the total concentration $n_0 = n_{\varepsilon \leqslant \varepsilon_c} + n_{\varepsilon > \varepsilon_c}$ of the excitons during the adiabatic relaxation yields the integral relationship

$$\frac{1}{\sqrt{\tau}} \int_0^1 \frac{du}{(1-u)^{3/2}} \left[e^{\rho(\tau) - \rho(u\tau)} - \frac{1}{\sqrt{u}} \right] = \zeta(3/2) \left[\left(\frac{T_c}{T_b} \right)^{3/2} - 1 \right]. \tag{9}$$

Equation (9), which is the functional integral equation for $\rho(\tau)$, attributes the final condensate fraction $n_c = n_0[1-(T_b/T_c)^{3/2}]$ to be spread in the nonequilibrium region $\varepsilon \leqslant \varepsilon_c(\tau) = e_c(\tau)/T_b \ll 1$.

In order to find $\rho(\tau)$ from Eq. (9), we use the ansatz

$$F(\tau)/F(u\tau) = A(u)\sqrt{\tau} + 1/\sqrt{u}, \tag{10}$$

where $F(\tau) = e^{\rho(\tau)}$ and $A(u)$ is a unknown function. Equation (10) provides a time-independence of the lhs of Eq. (9) in accordance with its rhs. For $u = 1/\tau$ in the interval $0 \leqslant u \leqslant 1$ and where Eqs. (6) and (9) are defined, Eq. (10) reduces to

$$A(1/\tau) = \sqrt{u\tau} A(u)[1 + A(1/u\tau)] + A(1/u\tau), \tag{11}$$

with the additional requirement $A(1) = 0$. The further expansion of the function A around the inverse point $1/\tau = u + \epsilon$, where ϵ is an infinitesimal value, leads to

$$A'(u) + \left[\frac{\alpha}{u} + \frac{1}{2u} \right] A(u) = -\frac{\alpha}{u}, \tag{12}$$

where $\alpha = -A'(1)$. In the derivation of Eq. (12) we keep only the leading terms $o(\epsilon)$ and neglect the smaller terms $o(\epsilon^2)$.

The solution of Eq. (12), which satisfies the condition $A(1)=0$, is given by

$$A(1/\tau)=\frac{\alpha}{\alpha+1}[\tau^{\alpha+1/2}-1]. \tag{13}$$

Here the unknown parameter α is determined through the substitution of the solution (13) in Eqs. (10) and (9) by

$$\left(\frac{\alpha}{\alpha+1}\right)\int_0^1\frac{du}{(1-u)^{3/2}}[u^{-\alpha-1/2}-1]=\zeta(3/2)\left[\left(\frac{T_c}{T_b}\right)^{3/2}-1\right]. \tag{14}$$

Finally, the function $\rho(\tau)$ is given by

$$\rho(\tau)=\ln\{[A(1/\tau)+1]\sqrt{\tau}F(0)\}, \tag{15}$$

where $A(1/\tau)$ is determined by Eq. (13) and $F(0)$ is a positive constant. This constant drops out the final expressions due to the combination $\rho(\tau)-\rho(u\tau)$ presented in the inhomogeneous solution $N_\varepsilon^{\rm inhom}(\tau)$ and Eq. (9). From Eqs. (8), (15), and (13) one gets the fundamental law of the phonon-assisted adiabatic cooling of the high-energy quasiequilibrium excitons:

$$\delta T(\tau)=\left(\frac{T_b^2}{e_0}\right)\frac{1}{2\tau}\left[1+\frac{1+2\alpha}{1+(1/2\alpha)\tau^{-\alpha-1/2}}\right]. \tag{16}$$

With this law, the nonequilibrium distribution of the low-energy excitons is determined by the inhomogeneous solution of Eq. (5) as

$$\begin{aligned}N_\varepsilon(\tau)&=\tau\int_0^1\left[\frac{\alpha\sqrt{\tau}}{1/2+\alpha}(u^{-\alpha-1/2}-1)+\frac{1}{\sqrt{u}}\right]e^{-\tau\varepsilon(1-u)}du\\&=\frac{\alpha\sqrt{\tau}}{1/2+\alpha}\left[\frac{\tau^{3/2}}{1/2-\alpha}e^{-\tau\varepsilon}\Phi(1/2-\alpha,3/2-\alpha,\tau\varepsilon)\right.\\&\quad\left.-\frac{\sqrt{\tau}}{\varepsilon}(1-e^{-\tau\varepsilon})\right]+2\sqrt{\frac{\tau}{\varepsilon}}Ds(\sqrt{\tau\varepsilon}),\end{aligned} \tag{17}$$

where Φ is the degenerate hypergeometric function [15] and $Ds(x)=\exp(-x^2)\int_0^x dt\,\exp(t^2)$ is Dawson's integral [16].

Equations (16) and (17) are the *generic solution* of the acoustic phonon-assisted Boltzmann kinetics of a Bose gas at $T_b\leqslant T_c$. This generic solution is completely defined by the ratio T_c/T_b of the critical and bath temperatures through the parameter α given by Eq. (14). Equation (14) has only one solution $0\leqslant\alpha<1/2$ for given T_c/T_b (see the inset of Fig. 2). For $T_b\to T_c$ ($T_b\leqslant T_c$) the solution $\alpha\ll 1/2$ can be approximated by

$$\alpha=\frac{1}{4}\zeta(3/2)\left[\left(\frac{T_c}{T_b}\right)^{3/2}-1\right]=\left(\frac{\hbar^2}{2\sqrt{\pi}M_xT_b}\right)n_c, \tag{18}$$

where $\alpha=0$ for $T_b=T_c$. However, for $T_b\to 0$ ($\alpha\to 1/2$), this simple approximation of the parameter α through the condensate fraction n_c becomes invalid.

The homogeneous solution $N_\varepsilon^{\rm hom}(\tau)$ of the reduced kinetic Eq. (5) with $\rho(\tau)$ of Eq. (15) is given by

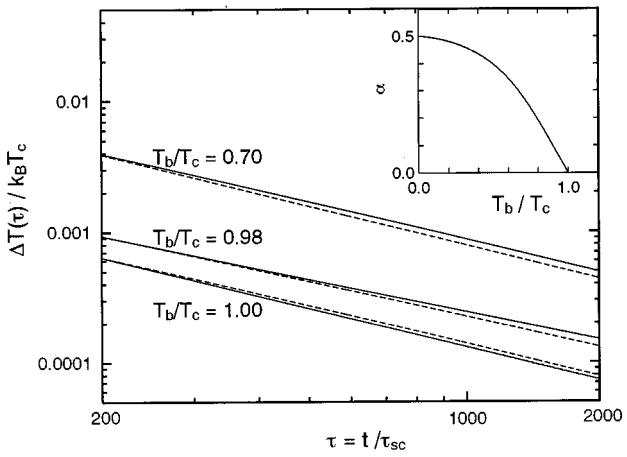

FIG. 2. Adiabatic cooling of the high-energy quasiequilibrium excitons. Solid lines, numerical evaluation of Eq. (2); dashed lines, generic law (16). Inset: $\alpha=\alpha(T_b/T_c)$ as a numerical solution of Eq. (14).

$$N_\varepsilon^{\rm hom}(\tau)=\widetilde{C}_\varepsilon\sqrt{\tau}\left[1+\frac{\alpha}{1/2+\alpha}(\tau^{1/2+\alpha}-1)\right], \tag{19}$$

where $\widetilde{C}_\varepsilon=C_\varepsilon F(0)\geqslant 0$ is the renormalized integration constant dependent on the initial distribution at $t=0$. The homogeneous contribution $N_\varepsilon^{\rm hom}(\tau)$ to the complete solution (6) of Eq. (5) decays with τ with respect to $N_\varepsilon^{\rm inhom}(\tau)$, as seen from the comparison of Eqs. (17) and (19). For example, for the ground-state mode one gets

$$\left.\frac{N_{\varepsilon=0}^{\rm hom}(\tau)}{N_{\varepsilon=0}^{\rm inhom}(\tau)}\right|_{\tau\to\infty}\propto\left(\frac{1/2-\alpha}{1/2+\alpha}\right)\left.\frac{1}{\tau^{1/2-\alpha}}\right|_{\tau\to\infty}\to 0. \tag{20}$$

Actually, the influence of the homogeneous solution $N_\varepsilon^{\rm hom}(\tau)$ on the total relaxation kinetics disappears already during the first transient stage. Therefore, from an arbitrary initial $N_e(t=0)$, the following time-dependent distribution of excitons approaches at $\tau\geqslant\tau_{\rm tr}$ the generic solution (17) of the adiabatic stage. The only information from an initial distribution, which influences the total kinetics at $T_b\leqslant T_c$, is the duration $\Delta t_{\rm tr}$ of the first transient stage, typically a few $\tau_{\rm sc}$.

III. DISCUSSION

In order to test the generic solution (16) and (17) we model the phonon-assisted relaxation of excitons within the initial kinetic Eq. (2) reduced to the energy space. A homogeneous grid with 1200 points for $\sqrt{\varepsilon}$ is used to cover a close vicinity of the ground-state mode $\varepsilon=0$ (the maximum value of the dimensionless energy is $\varepsilon_{\rm max}=20$). Equation (2) is evaluated by a fourth-order Runge-Kutta integration routine with the time step 0.01–$0.05\tau_{\rm sc}$. The calculations of the phonon-assisted kinetics given in Figs. 2–4 take about 20 hours on an IBM Risk-6000 workstation.

For the numerical simulations we use the parameters of para-excitons in Cu_2O [17]. *The numerical evaluations* within Eq. (2) of the relaxation kinetics at $T_b\leqslant T_c$ for various initial distributions $N_\varepsilon(\tau=0)$ *completely confirm the analytic analysis*, namely, (i) the initial distribution of excitons

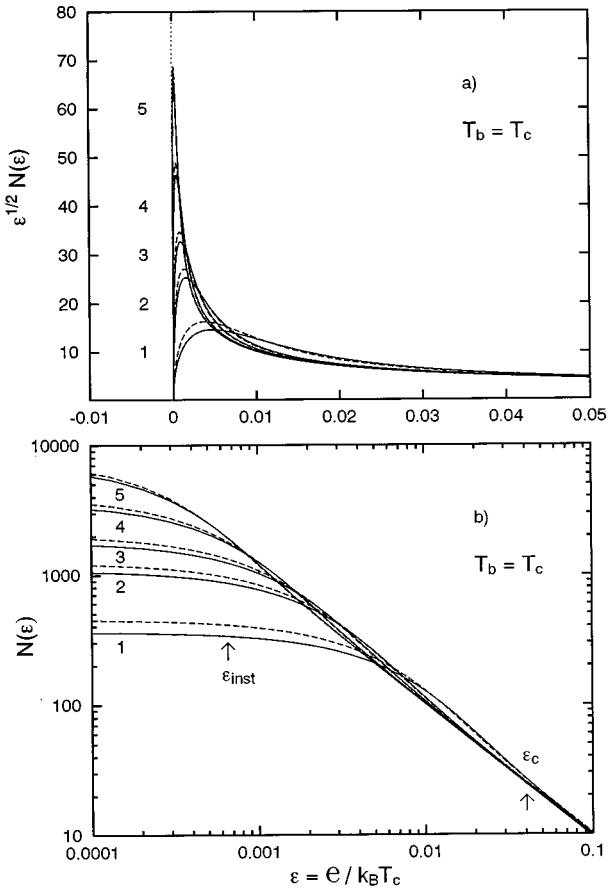

FIG. 3. Evolution of the distributions (a) $\varepsilon^{1/2}N_\varepsilon(\tau)$ and (b) $N_\varepsilon(\tau)$ for the adiabatic stage (dimensionless units, $T_c=T_b$). Solid lines, numerical evaluation of Eq. (2); dashed lines, generic solution (17): τ=194 (1), 582 (2), 971 (3), 1942 (4), and 3884 (5).

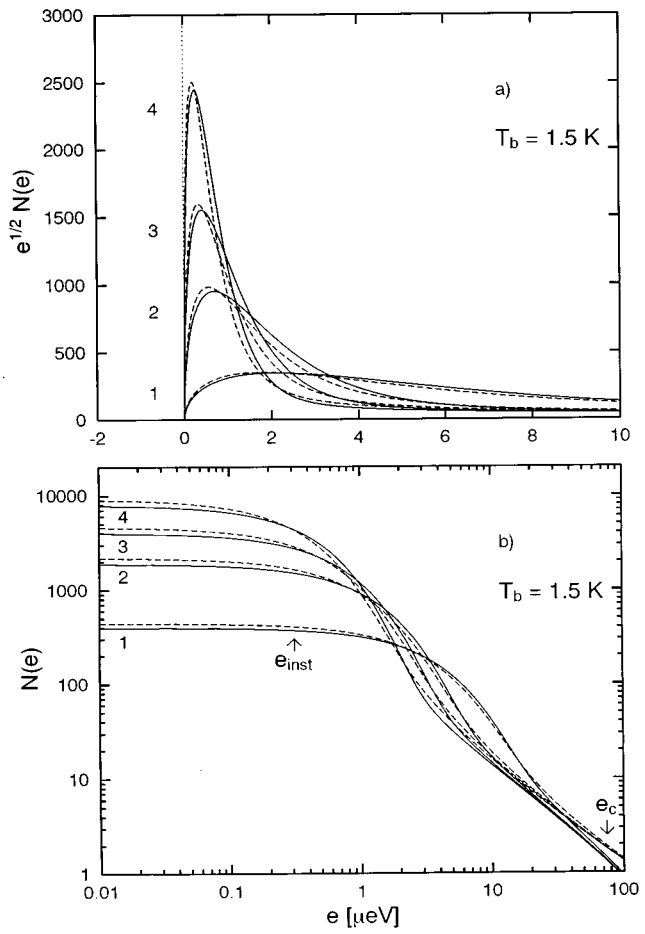

FIG. 4. Evolution of the distributions (a) $e^{1/2}N_e(t)$ and (b) $N_e(t)$ for the adiabatic stage at $T_b=1.5$ K$<T_c=2.135$ K (paraexcitons in Cu_2O, dimensional units, $\tau_{sc}=22.1$ ns). Solid lines, numerical evaluation of Eq. (2); dashed lines, generic solution (17): $t=1$ μs (1), 3 μs (2), 5 μs (3), and 8 μs (4).

decays within a time $\tau_{tr}=\Delta t_{tr}/\tau_{sc}$ of a few scattering processes; (ii) after the initial transient kinetics, the high-energy excitons with $\varepsilon \geqslant \varepsilon_c(\tau \geqslant \tau_{tr})$ quasiequilibrate with an effective $\mu=0$ and $0<\delta T/T_b \ll 1$; (iii) for $\tau > \tau_{tr}$, the effective temperature $T=T_b+\delta T(\tau)$ and the distribution function of the low-energy nonequilibrium excitons ($\varepsilon \leqslant \varepsilon_c$) asymptotically approach the generic solution (16) and (17), respectively, with α given by Eq. (14).

The time dependence $\delta T=\delta T(\tau)$ of the adiabatic approach of the effective temperature $T=T_b+\delta T(\tau)$ to the bath one are compared in Fig. 2 with the generic solution (16) for various $T_b \ll T_c$. According to the numerical evaluations of Eq. (2), the distribution of high-energy excitons indeed follows the Planck function with $T=T_b+\delta T(\tau)$ for $\varepsilon \geqslant \varepsilon_c(\tau)$. In Figs. 3(a) and 3(b) the time evolution of the distributions $\sqrt{\varepsilon}N_\varepsilon(\tau)$ and $N_\varepsilon(\tau)$ as a numerical solution of Eq. (2) is compared with the corresponding generic solution (17) for $T_b=T_c$. The similar comparison for $T_b<T_c$ is given in Figs. 4(a) and 4(b), where we use the dimensional energy e and time t for paraexcitons in Cu_2O. Again, Figs. 3 and 4 show that the derived fundamental solution (16) and (17) with α given by Eq. (14) reproduces quite well the adiabatic stage of the phonon-assisted relaxation kinetics. Moreover, we have also checked that the adiabatic stage is insensitive to the initial distribution at $t=0$. The initial $N_e(t=0)$ determines only the duration Δt_{tr} of the first transient in the relaxation kinetics.

The generic equation (17) yields the following adiabatic kinetics for the ground-state mode $\varepsilon=0$:

$$N_{\varepsilon=0}(\tau)=2\tau+\left(\frac{\alpha}{1/2-\alpha}\right)\tau^{3/2}. \tag{21}$$

The *nonexponential* relaxation into the ground-state mode results in a strong slowing down of the entire phonon-assisted kinetics. Furthermore, from Eqs. (16) and (17) one concludes the marginal character of relaxation kinetics at $T_b=T_c$ ($\alpha=0$),

$$\delta T(\tau)=\left(\frac{T_b^2}{e_0}\right)\frac{1}{2\tau}, \quad N_{\varepsilon=0}(\tau)=2\tau, \tag{22}$$

while for $T_b<T_c$ ($0<\alpha<1/2$)

$$\delta T(\tau)\Big|_{\tau\to\infty}\to\left(\frac{T_b^2}{e_0}\right)\frac{(1+\alpha)}{\tau},$$

$$N_{\varepsilon=0}(\tau)_{\tau\to\infty}\to\left(\frac{\alpha}{1/2-\alpha}\right)\tau^{3/2}. \tag{23}$$

According to Eqs. (22) and (23), the coefficient in asymptotics of $\delta T(\tau\to\infty)$ has a jump at $\alpha=0$ $(T_b=T_c)$. Moreover, the relaxation kinetics into the ground-state mode follows the different power laws at $T_b=T_c$ and $T_b<T_c$. Both the spontaneous and the stimulated relaxation [the first and the second term on the rhs of Eq. (4)] contribute equally proportional to τ to the ground-state kinetics (22) at $T_b=T_c$. In contrast, only the stimulated kinetics is responsible for $N_{\varepsilon=0}\propto\tau^{3/2}$ at $T_b<T_c$. For $T_b=T_c$ the temperature law (22) is identical to that which characterizes a critical slowing down of the thermodynamic relaxation if $T_b\to T_c$ from above $(T_b>T_c)$. This critical slowing down of the thermodynamic processes is a general feature of second-order phase transitions [18].

According to Eq. (5), there is an instability kernel $\varepsilon\leqslant\varepsilon_{\rm inst}(\tau)=(e_0/T_b^2)\delta T(\tau)\ll\varepsilon_c(\tau)$, where the stimulated kinetics increases the occupation numbers [positive sign of the second term on the rhs of Eq. (5)]. In contrast, the stimulated processes depopulate the states in the energy range $\varepsilon_{\rm inst}(\tau)<\varepsilon\leqslant\varepsilon_c(\tau)$. From Eq. (17) one derives $\varepsilon_c(\tau)=[1/\tau+\alpha/\sqrt{\tau}]/\kappa$, where the dimensionless parameter $0<\kappa<1$ characterizes the relative deviation of N_{ε_c} from the equilibrium value $N_{\varepsilon_c}^{\rm eq}$ at $\varepsilon=\varepsilon_c$ and $\tau\kappa\gg1$. The collision integral of the Boltzmann kinetics contains the energy conservation $\delta(e_{\mathbf{k}}-e_{\mathbf{p}}\pm\hbar\omega_{\mathbf{p}-\mathbf{k}})$ as a quasiclassical approximation. This approximation fails for the very low-energy excitons $[e\leqslant e_B(t)]$, where $te_B(t)\leqslant\hbar$ (see the corresponding discussion in Ref. [19]). From this time-energy uncertainty, which breaks the classical Boltzmann kinetics for $\varepsilon<\varepsilon_B=e_B/T_b$, we find $\varepsilon_B(\tau)=\hbar/\tau_{\rm sc}(T_b)T_b\tau$. According to our estimates, $\varepsilon_B(\tau)\ll\varepsilon_{\rm inst}(\tau)$, e.g., $\varepsilon_B\simeq0.01\varepsilon_{\rm inst}$ for the numerical calculations of Figs. 2–4. For $(T_c-T_b)/T_c\ll1$ the three critical boundaries $\varepsilon_B(\tau)\ll\varepsilon_{\rm inst}(\tau)\ll\varepsilon_c(\tau)$ shrink in parallel, following the law (16) of the adiabatic cooling.

The final equilibrium distribution at $T_b\leqslant T_c$ is given by $N_\varepsilon^{\rm eq}=(n_c/\sqrt{\varepsilon})\,\delta(\varepsilon)+1/\varepsilon$ for $\varepsilon\ll1$. The last term on the rhs of the generic solution (17) yields the noncondensate Planck distribution at $0<\varepsilon\ll1$, i.e., $2(\tau/\varepsilon)^{1/2}Ds(\sqrt{\tau\varepsilon})\to1/\varepsilon$ for $\tau\varepsilon\to\infty$. On the other hand, the first term proportional to α on the rhs of Eq. (17) describes the time evolution of the condensate fraction n_c towards the ground-state mode $\varepsilon=0$. For example, for $0\leqslant\alpha\ll1/2$ and $\tau\varepsilon\to\infty$ one gets from Eq. (17)

$$N_\varepsilon(\tau\varepsilon\to\infty)=\left(\frac{\alpha\sqrt{\pi}}{\sqrt{\varepsilon}}\right)I(\tau\varepsilon,\varepsilon)+1/\varepsilon, \tag{24}$$

where

$$I(\tau\varepsilon,\varepsilon)=\frac{1}{\varepsilon(\pi\tau\varepsilon)^{1/2}}[1-(\tau\varepsilon+1)e^{-\tau\varepsilon}] \tag{25}$$

is the δ-like function with a continuously decreasing width proportional to $1/\tau\varepsilon$ and $\int_0^\infty I(\tau\varepsilon,\varepsilon)d\varepsilon=1$. Therefore, from Eqs. (24), (25), and (18) we conclude that the generic solution (17) indeed evolves to the true equilibrium distribution $N_\varepsilon^{\rm eq}$. Because the Dawson integral in Eq. (17) is responsible for the equilibrium distribution $1/\varepsilon$ at $\tau\varepsilon\to\infty$, in order to adapt the fundamental solution to an arbitrary dimensionless energy ε, including $\varepsilon\geqslant1$, we should replace ε by $e^\varepsilon-1$ in the last term of Eq. (17).

Although the numerical results of Figs. 2–4 are adapted to the paraexcitons in Cu_2O, the fundamental solution (16) and (17) refers to the degenerate phonon-assisted Boltzmann kinetics of an arbitrary three-dimensional gas of ideal bosons with quadratic dispersion. According to Eq. (17), $\sqrt{\varepsilon}N_\varepsilon\to0$ for $\varepsilon\to0$ and any given $\tau>0$ [see also Figs. 3(a) and 4(a)], while $\sqrt{\varepsilon}N_\varepsilon^{\rm eq}\propto1/\sqrt{\varepsilon}\to\infty$ for $\varepsilon\to0$. As a result, BEC into the ground-state mode builds up within the phonon-assisted kinetics only at $\tau\to\infty$. This result is consistent with conclusions of Ref. [4]. Moreover, a genuine Bose-Einstein condensate, which possesses such a nonlinear property as *coherence*, finally develops due to the conservative exciton-exciton interaction Ref. [20]. Even for an arbitrary small but finite, exciton-exciton interaction U_0, a coherent region $\varepsilon<\varepsilon_{\rm coh}$ arises due to the strong accumulation of excitons at the close vicinity of $\varepsilon=0$. This coherent region, where the potential exciton-exciton interaction exceeds the corresponding kinetic energies $e_{\mathbf{k}}$ of excitons, cannot be treated within kinetic equations [20]. We can estimate a rise of the coherent stage by $U_0\Sigma_{\varepsilon\leqslant\varepsilon_{\rm coh}}N_\varepsilon(\tau)\geqslant\varepsilon_{\rm coh}(\tau)$ as a condition for $\varepsilon_{\rm coh}(\tau)$ and $\tau=\tau_{\rm coh}$. For the equilibrium time-independent distributions $N_\varepsilon^{\rm eq}$ this criterion indeed holds only for $T_b\leqslant T_c$ when $\mu=0$, i.e., the coherent region arises simultaneously with the macro-occupation of the ground-state mode.

For excitons in Cu_2O the exciton-exciton interaction can be estimated as $U_0=4\pi\hbar^2a_x/M_x\simeq2.8\times10^{-22}$ eV cm^3, where $a_x\simeq7$ Å is the exciton Bohr radius. According to the numerical calculations, the criterion yields $\tau_{\rm coh}\simeq2000-3000$ scattering times for the rise of the coherent stage. Our analysis of the phonon-assisted kinetics at $T_b\leqslant T_c$ is valid only for $\tau\leqslant\tau_{\rm coh}$. However, the derived fundamental solution is also important as the initial condition for the coherent stage. The mutual synchronization of modes $\varepsilon\leqslant\varepsilon_{\rm coh}(\tau)$ breaks the random-phase approximation and leads to coherence. This conservative process does not involve modes $\varepsilon>\varepsilon_{\rm coh}$ [19,20].

The generic solution for the adiabatic relaxation due to the exciton-exciton interaction is an intriguing question. Recent numerical simulations [4,21] of the approach to equilibrium at $T\leqslant T_c$ through the exciton-exciton scattering give some hints on the existence of such a solution. However, in this case both the initial transient and the following adiabatic stage should be considerably shorter than for the considered phonon-assisted kinetics because the four-particle exciton-exciton interaction is more effective than the three-particle exciton-phonon coupling.

The exciton-exciton (boson-boson) relaxation kinetics in the presence of a fermion bath has been examined in detail in Ref. [19]. Although a critical slowing down (adiabatic stage) has been found for $T_b<T_c$, the derived solution is not a

generic one. This solution traces a kinetic evolution from the particular initial Planck distribution $N_e(t=0)=1/\exp(e/T_c)-1$, i.e., is a homogeneous solution of the Boltzmann kinetics. However, a true generic solution should be independent of the initial distribution and refers to an inhomogeneous solution of the exciton-exciton Boltzmann equation.

IV. CONCLUSION

In this work we develop the theory and evaluate numerically the phonon-assisted Boltzmann kinetics of a degenerate Bose gas. The following conclusions summarize our study.

(i) For $T_b \leqslant T_c$, the phonon-assisted relaxation kinetics is given by the following scenario: Within a few scattering times τ_{sc}, an initial distribution of Bose particles disappears and the relaxation towards a quantum degenerate equilibrium state with a Bose-Einstein condensate is a slow adiabatic process. This second kinetic stage is described uniquely by the generic solutions (16) and (17). This result has a rather general form and can be applied to a system of any bosons with quadratic dispersion provided that relaxation kinetics is determined by a three-particle interaction: boson plus bath excitation$\leftrightarrow$boson. For example, the fundamental solutions (16) and (17) can be easily adapted to the Fröhlich exciton–LA-phonon interaction in polar semiconductors ($M_{x\text{-ph}} \propto |\mathbf{p}-\mathbf{k}|$) or to LO–phonon-assisted relaxation kinetics.

(ii) Our straightforward numerical modeling of the phonon-assisted Boltzmann kinetics at $T_b \leqslant T_c$ provides evidence for the proposed relaxation picture and the generic solutions (16) and (17).

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